2020.10.1 P30 / 4.2.3

$K = \mathfrak{B}(V)^{\mathrm{co}\,\mathfrak{B}(V_1)} \simeq \mathfrak{B}(K')$ $\mathrm{ad}_c\, x_1$

$K' = \mathrm{ad}_c\, \mathfrak{B}(V_1)(V_2)$ cf. 4.1.4

ghost: $\mathfrak{g} = \begin{cases} -2a, & \epsilon = 1 \\ a, & \epsilon = -1 \end{cases}$

Rmk 4.10 If $\mathfrak{g}$ is discrete, then the family $(z_n)_{0 \le n \le 2|a|}$ is a basis of K' $a = q_{21}^{-1} \vartheta(g_2)$

$(z_n := (\mathrm{ad}_c x_2)^n x_3)$

Pf: ① z_n's are homogeneous of distinct degrees, and are $\neq 0$ by (4.13)

(4.13: $\partial_3(z_{2k}) = \mu_{2k} x_{21}^k$, $\partial_3(z_{2k+1}) = \mu_{2k+1} x_1 x_{21}^k$)

$\partial_1(z_{2k}) = 0$ Lem 4.8

$(z_0 = x_3,\ z_1 = (\mathrm{ad}_c x_2) x_3 = x_2 x_3 - \mu c(x_2 \otimes x_3))$
$= x_2 x_3 - (g_1 \cdot x_3) x_2$

$V_1 \leftarrow x_1, x_2$

$V_2 \leftarrow x_3$

$(c(x_i \otimes x_j))_{i,j \in \mathbb{I}_2} = \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 \end{pmatrix}$

$a = q_{21}^{-1} \vartheta(g_2)$ $c^2(x_1$

$\mathfrak{g} = \begin{cases} -2a, \\ a \end{cases}$

② $\forall n \in \mathbb{N}_0$

(4.15) $\mathrm{ad}_c x_1 (z_n) = x_1 z_n - (g_1 \cdot z_n) x_1 \overset{(4.8)}{=} \epsilon^n q_{12} z_n x_1 - \epsilon^n q_{12} z_n x_1 = 0$

不需要? (4.16) $\mathrm{ad}_c x_{21}(z_n) = \mathrm{ad}_c x_2\, \mathrm{ad}_c x_1 (z_n) - \epsilon\, \mathrm{ad}_c x_1\, \mathrm{ad}_c x_2 (z_n) = 0$

$\because$ P27 / (4.8) $g_1 \cdot z_n = \epsilon^n q_{12} z_n$ $V_1 = \mathcal{V}_{g_1}(\chi_1, \vartheta)$, $V_2 = k_{g_2}^{\chi_2}$

$x_{21} = (\mathrm{ad}_c x_2) x_1$ $\epsilon = \pm 1$

$\therefore K'$ is generated by z_n. □

Defn: $\epsilon = 1$, Then $\nu_{n,n} = 1$

$\nu_{0,n+1} = -(\frac{n}{2} + a)\nu_{0,n}$, $\nu_{k,n+1} = \nu_{k-1,n} - (\frac{n+k}{2} + a)\nu_{k,n}$, $1 \le k \le n$

Lemma 4.11 $K' \in$
$\delta = (\pi_{\mathfrak{B}(V_1) \# k\Gamma} \otimes \mathrm{id})\Delta_{\mathfrak{B}(V) \# k\Gamma}$

If $\epsilon = 1$, δ on z_n: $\pi: \mathfrak{B}(V_1) \# k\Gamma$, $x_3 \mapsto 0$

(4.17) $\delta(z_n) = \sum_{k=0}^{n} \nu_{k,n} x_1^{n-k} g_1^k g_2 \otimes z_k$

If $\epsilon = -1$, δ on z_n:

(4.18) $\delta(z_{2n}) = \sum_{k=1}^{n} k \binom{n}{k} \mu_{k,n} x_1 x_{21}^{n-k} g_1^{2k-1} g_2 \otimes z_{2k-1} + \sum_{k=0}^{n} \binom{n}{k} \mu_{k,n} x_{21}^{n-k} g_1^{2k} g_2 \otimes z_{2k}$

(4.19) $\delta(z_{2n+1}) = \sum_{k=0}^{n} \binom{n}{k} \mu_{k,n+1} x_1 x_{21}^{n-k} g_1^{2k} g_2 \otimes z_{2k} + \sum_{k=0}^{n} \binom{n}{k} \mu_{k+1,n+1} x_{21}^{n-k} g_1^{2k+1} g_2 \otimes z_{2k+1}$

Pf: n: induction

when $n = 0$, $\delta(z_0) = \delta(x_3) = g_2 \otimes z_0$ $(z_0 = x_3 \in k_{g_2}^{\chi_2})$

① suppose $\epsilon = 1$, (4.17)?

$\delta(z_{n+1}) = \delta(x_2 z_n - \epsilon^n q_{12} z_n x_2)$

$\overset{(4.10)}{=} (x_2 \otimes 1 + g_1 \otimes x_2)\delta(z_n) - q_{12}\delta(z_n)(x_2 \otimes 1 + g_1 \otimes x_2)$

$\delta(x_2) = x_2 \otimes 1 + g_1 \otimes x_2$ $V_1 = \mathcal{V}_{g_1}(\chi_1, \vartheta)$

$= \sum_{k=0}^{n} \nu_{k,n} (x_2 x_1^{n-k} g_1^k g_2 \otimes z_k + x_1^{n-k} g_1^{k+1} g_2 \otimes x_2 z_k - q_{12} x_1^{n-k} g_1^k g_2 x_2 \otimes z_k - q_{12} x_1^{n-k} g_1^{k+1} g_2 \otimes z_k x_2)$

? $\overset{(*)}{=} -\sum_{k=0}^{n} (\frac{n+k}{2} + a)\nu_{k,n} x_1^{n+1-k} g_1^k g_2 \otimes z_k + \sum_{k=1}^{n+1} \nu_{k-1,n} x_1^{n+1-k} g_1^k g_2 \otimes z_k$

(*) (3.5) $x_2 x_1^n = x_1^n x_2 - \frac{n}{2} x_1^{n+1}$, $x_2 g_1^k = g_1^k x_2 - k x_1 g_1^k$

$= -(\frac{n}{2} + a)\nu_{0,n} x_1^{n+1} g_2 \otimes z_0 + \sum_{k=1}^{n} [\nu_{k-1,n} - (\frac{n+k}{2} + a)\nu_{k,n}] x_1^{n+1-k} g_1^k g_2 \otimes z_k$
$+ \nu_{n,n} g_1^{n+1} g_2 \otimes z_{n+1}$

$= \nu_{0,n+1} x_1^{n+1} g_2 \otimes z_0 + \sum_{k=1}^{n} \nu_{k,n+1} x_1^{n+1-k} g_1^k g_2 \otimes z_k + \nu_{n+1,n+1} g_1^{n+1} g_2 \otimes z_{n+1}$

$= \sum_{k=0} x_1^{n-k} g_1^k g_2 \otimes z_k$

② $\epsilon = -1$, (4.19), n ✓

$\delta(z_{2n+2}) \overset{(4.10)}{=} (x_2 \otimes 1 + g_1 \otimes x_2)\delta(z_{2n+1}) - q_{12}\delta(z_{2n+1})(x_2 \otimes 1 + g_1 \otimes x_2)$

$= \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k,n+1} x_2 x_1 x_{21}^{n-k} g_1^{2k} g_2 \otimes z_{2k}}_{A} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k+1,n+1} x_2 x_{21}^{n-k} g_1^{2k+1} g_2 \otimes z_{2k+1}}_{B}$

$+ \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k,n+1} g_1 x_1 x_{21}^{n-k} g_1^{2k} g_2 \otimes x_2 z_{2k}}_{C} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k+1,n+1} g_1 x_{21}^{n-k} g_1^{2k+1} g_2 \otimes z_{2k+1}}_{D}$

$- q_{12} \underbrace{(\sum_{k=0}^{n} \binom{n}{k} \mu_{k,n+1} x_1 x_{21}^{n-k} g_1^{2k} g_2 x_2 \otimes z_{2k}}_{E}$

$+ \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k+1,n+1} x_{21}^{n-k} g_1^{2k+1} g_2 x_2 \otimes z_{2k+1}}_{F}$

$+ \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k,n+1} x_1 x_{21}^{n-k} g_1^{2k} g_2 g_1 \otimes z_{2k} x_2}_{G}$

$+ \underbrace{\sum_{k=0}^{n} \binom{n}{k} \mu_{k+1,n+1} x_{21}^{n-k} g_1^{2k+1} g_2 g_1 \otimes z_{2k+1} x_2}_{H}$

令 $A - q_{12} E = \sum_{k=0}^{n} \binom{n}{k} \mu_{k,n+1} I \otimes z_{2k}$

where $I := x_2 x_1 x_{21}^{n-k} g_1^{2k} g_2 - x_1 x_{21}^{n-k} g_1^{2k} g_2 x_2$

$\overset{3.7, 4.1}{=} x_{21}^{n-k+1} g_1^{2k} g_2 - x_1 x_2 x_{21}^{n-k} g_1^{2k} g_2 - x_1 x_{21}^{n-k+1} g_1^{2k} (x_2 + a x_1) g_2$

$= (x_{21}^{n+1-k} - x_1 x_2 x_{21}^{n-k} - x_1 x_2^{n-k}(-x_2 + (2k - a)x_1)) g_1^{2k} g_2$

(3.10)(4.9)

$\overset{(3.7)}{=} x_{21}^{n+1-k} g_1^{2k} g_2$

$B - q_{12} F$, $D - q_{12} H$, $C - q_{12} G$

$(B + C) - q_{12}(F + G)$
$(A + D) - q_{12}(E + H)$

Thm 4.12 GK dim $\mathfrak{B}(V)$ is finite, iff ϵ, q_{22} and $\mathfrak{g}$ are as in Table 6

weak interaction ($q_{12} q_{21} = 1$)

ϵ	q_{22}	$\mathfrak{g}$	GKdim K
1	1	dis…	$\mathfrak{g}+1$
	−	dis…	0
	$\in G_3'$	1	2
−1	1	dis…	$\mathfrak{g}+1$
	·1	dis…	$\mathfrak{g}$

Pf: by Lem 4.9 (P29), Suppose $\mathfrak{g}$ is discrete

Claim: the braid V.S. K' is of diagonal type with braiding matrix
$(p_{ij})_{0 \le i,j \le 2|a|} = (\epsilon^{ij} q_{12}^i q_{21}^j q_{22})_{0 \le i,j \le 2|a|}$

more: $p_{ii} = \epsilon^i q_{22}$, $p_{ij} p_{ji} = q_{22}^2$ $\forall k\ j \in \{0\} \cup \mathbb{I}_{2|a|}$

Pf of Claim: 由 Rmk 4.10

$c(z_i \otimes z_j) \overset{?}{=} g_1^i g_2 \cdot z_j \otimes z_i$

$c((\mathrm{ad}_c x_2)^i x_3 \otimes (\mathrm{ad}_c x_2)^j x_3)$
$= (\mathrm{ad}_c x_2)^j x_3 \otimes$ —
$= g_1^i g_2 \cdot z_j \otimes z_i$

$\overset{(4.9)}{=} g_1^i \cdot q_{21}^j q_{22}\, z_j \otimes z_i$

$\overset{(4.8)}{=} \epsilon^{ij} q_{12}^i q_{21}^j q_{22}\, z_j \otimes z_i$

claim □

Case 1. $q_{22}^2 = 1$

K' $\epsilon^i q_{22}$

p_{11} p_{22} p_{33} 不存在线

Case 2 $\epsilon = 1$, $q_{22} \in G_3'$, $\mathfrak{g} = 1$

Dynkin A_2

$\therefore \mathfrak{B}(K')$ f.d. $\therefore$ GK-dim $= 0$

Case 3. $\epsilon = 1$, $q_{22}^2 \neq 1$, $q_{22} \notin G_3'$

$\because \mathfrak{g} \ge 1$, 2

The Dynkin subdiagram -- vertices 0 and 1

labels q_{22} and q_{22}^2 on the edge? $\tilde{q}_{ij} = p_{ij} p_{ji}$

① If $q_{22} \notin G_\infty$, then K' doesn't admit all reflections (参考文献 3)

hence GKdim $\mathfrak{B}(K') = \infty$

② If $q_{22} \in G_M'$, with $M \ge 4$

$\begin{pmatrix} 2 & 2-M \\ 2-M & 2 \end{pmatrix}$

then GKdim $\mathfrak{B}(K') = \infty$, by Thm 1.6

Case 4. $\epsilon = 1$, $q_{22} \in G_3'$, $\mathfrak{g} \ge 1$

$A_2^{(1)}$ GK — $= \infty$

Case 5. $\epsilon = -1$, $q_{22}^2 \neq 1$

$\because a \neq 0$ K' 3 vertices

① q_{22} isn't a root of 1.

GK — $= \infty$

② $q_{22} \in G_N'$ $N \ge 4$ $\begin{pmatrix} 2 & 2-N \\ 2-N & 2 \end{pmatrix}$

GKdim $= \infty$

③ $q_{22} \in G_3'$ $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

by Thm 1.6 GK — ∞

GKdim $\mathfrak{B}(V) = $ GKdim $K + 2$

$\mathfrak{B}(V) \simeq \mathfrak{B}(K') \# \mathfrak{B}(V_1)$ Lem 2.2, Prop 3.4

P27 / Rmk 4.8